IE532 Stochastic Models

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Basic Proof Techniques

- Proof by Construction

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- Proof by Contradiction

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- Proof by Contradiction
- Proof by Induction

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- a.k.a. direct proof

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- Simplest and easiest method
- Not very helpful in advanced topics

Proof by Construction

Theorem

If a and b are consecutive integers, then the sum $a + b$ is odd.

Proof.

Assume that a and b are consecutive integers. Because a and b are consecutive we know that $b = a + 1$ and $a + b = 2a + 1$. Thus, there exists a number k such that $a + b = 2k + 1$, so the sum $a + b$ is odd. $\square$

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Each proof ends with the phrase "proof is complete" or $\square$ or **QED** which stands for *quod erat demonstrandum* (what was to be demonstrated/shown).

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- Use these two facts to demonstrate a *contradiction*

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Assume that a and b are consecutive integers. Assume also that the sum $a+b$ is not odd. When $a+b$ is even, it can be represented as $2k$, $k \in \mathbb{Z}$. However, the integers a and b are consecutive, meaning $b = a+1$. Thus, we have derived that $2k = a + a + 1$, and also that $a = k - \frac{1}{2}$. As both a and k should be integers, this is a contradiction. $\square$

Proof by Induction

Here we consider the statement to be proven in recursive form and

- Show that a propositional form $P(x)$ is true for some basis case (usually when $x = 1$).
- **Assume** that $P(n)$ is true for some n
- **Show** that $P(n+1)$ is true.
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The sum $1 + 2 = 3$ is odd. Thus, the statement is true when $a = 1$. Assume that the statement is true for some a and b. This means, for some consecutive a and b we have that $a + b$ is odd. Next, we **have to** prove that the statement is true for $a + 1$ and $b + 1$. Sum of these two integers gives $a + 1 + b + 1 = a + b + 2$. $a + b$ is known to be odd (by previous assumption), thus $a + b + 2$ should also be odd as adding two to any integer does not change that integer's evenness or oddness. $\square$

Exercise

Prove that $\sqrt{2}$ is irrational.

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For each event E of the sample space S, a number $P(E)$ is defined as the probability of the event E if it satisfies the following three conditions:

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Intuitively, if an experiment is repeated over and over again then (with probability 1) the proportion of time that event E occurs will just be $P(E)$.

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Also note that

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

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Suppose we toss two dice. Let $E =$ Sum is 3, $F =$ First die is 4. Are these *independent*?
Suppose we toss two dice. Let $E =$ Sum is 7, $F =$ First die is 4. Are these *independent*?

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$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \ldots + P(E|F_n)P(F_n)$$

if F_i's are mutually exclusive and their union is the sample set.

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$$= \frac{P(W|H)P(H)}{P(W|H)P(H) + P(W|H^c)P(H^c)}$$

Bayes' Formula

Example: Consider two urns. The first contains two white and seven black balls, and the second contains five white and six black balls. We flip a fair coin and then draw a ball from the first urn or the second urn depending on whether the outcome was heads or tails. What is the conditional probability that the outcome of the toss was heads given that a white ball was selected?

Bayes' Formula

Example: Suppose that 5 percent of men and 0.25 percent of women are color-blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females.

Homework Assignment

1. Atiba and Ozan go shooting together. Both shoot at a target at the same time. Suppose Atiba hits the target with probability 0.9, whereas Ozan, independently, hits the target with probability 0.2.
   a. Given that exactly one of them hits the target, what is the probability that it was Ozan?
   b. Given that the target is hit, what is the probability that Ozan hits it?
2. Urn 1 has 1 white and 3 black balls. Urn 2 has 3 white and 8 black balls. We flip a fair coin. If the outcome is heads, then a ball from urn 1 is selected, while if the outcome is tails, then a ball from urn 2 is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails?
3. A Russian gangster kidnaps you. He puts two bullets in consecutive order in an empty six-round revolver, spins it, points it at your head and shoots. Turns out that it was empty, you're still alive. He then asks you, do you want me to spin it again and fire or pull the trigger again. For each option, what is the probability that you'll be shot?

Homework Assignment

4. Three prisoners are informed by their jailer that one of them has been chosen at random to be executed, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from $1/3$ to $1/2$, since he would then be one of two prisoners. What do you think of the jailer's reasoning?
5. Prove that $P(A|B) = P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$
6. An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn c additional balls of the same color are put in with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is $b/(b + r + c)$.

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Properties of cdf:

- $F(b)$ is a nondecreasing function of b,
- $\lim_{b\to\infty} F(b) = F(\infty) = 1$,
- $\lim_{b\to-\infty} F(b) = F(-\infty) = 0$.

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$$P\{X = a\} = ?$$

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The density is the derivative of the cumulative distribution function. A somewhat more intuitive interpretation may be obtained using

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Some Random Variables

- Discrete Random Variables
  - Bernoulli Random Variable (must know)
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  - Gamma Random Variable (only know sum of iid exponential RVs)
  - Normal Random Variable (know the shape)

Expectation of a Random Variable

- Discrete Case

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

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More on that (expectation of product is product of expectation when independent) later

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Discrete Random Variables - Summary

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p $0 \leqslant p \leqslant 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \leqslant p \leqslant 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Continuous Random Variables - Summary

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{e^{tb} - e^{ta}}{t(b-a)}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\dfrac{\lambda}{\lambda - t}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Gamma with parameters (n, λ) $\lambda > 0$	$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$	$\left(\dfrac{\lambda}{\lambda - t}\right)^n$	$\dfrac{n}{\lambda}$	$\dfrac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \dfrac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

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because

$$P\{X \in A\} = P\{X \in A, Y \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} \int_A f(x, y) dx dy = \int_A f_X(x) dx$$

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$E[g(X,Y)]$

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Ex: At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Find the expected number of men who select their own hats.

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$$\begin{aligned} Cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

Markov's Inequality

If X is a random variable that takes only nonnegative values, then for any value $a > 0$,

$$P\{X \geq a\} \leq E[X]$$

Chebyshev's Inequality

If X is a random variable with mean μ and variance σ^2, then for any value $k > 0$,

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

Strong Law of Large Numbers

Let $X_1, X_2, \ldots$ be a sequence of independent random variables having a common distribution, and let $E[X_i] = \mu$. Then, with probability 1,

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to \mu \quad \text{as } n \to \infty$$

Central Limit Theorem

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables, each with mean μ and variance σ^2. Then the distribution of

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is,

$$P\left\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$$

as $n \to \infty$.

Homework Assignment

1. An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

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3. If the density function of X equals $f(x) = \begin{cases} ce^{-2x} & 0 < x < \infty \\ 0 & x < 0 \end{cases}$

   Find c. What is $P\{X > 2\}$?

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4. Suppose that the joint probability density of X and Y is $f_{X,Y}(x,y) = \lambda^2 e^{-\lambda y}, 0 < x < y < \infty$. Find the density of X. Find the density of Y.

Homework Assignment

5. Let X represent the difference between the number of heads and the number of tails obtained when a fair coin is tossed n times. Define the probability mass function for X.
6. Prove that $E[X^2] \geq (E[X])^2$. When do we have equality?
7. Consider n independent flips of a coin having probability p of landing heads. Say a changeover occurs whenever an outcome differs from the one preceding it. For instance, if the results of the flips are H H T H T H H T, then there are a total of five changeovers. If $p = 1/2$, what is the probability there are k changeovers? What if $p < 1/2$? What is the probability there are k changeovers in n consecutive die rolls?
8. Let c be a constant. Show that $Var(cX) = c^2 Var(X)$ and $Var(c + X) = Var(X)$.

Conditional Probability and Expectation - Discrete

Consider discrete random variables X and Y

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If X is independent of Y, then $p_{X|Y}(x|y) = P(X = x)$

Conditional Probability and Expectation - Discrete

Example

There are n components. On a rainy day, component i will function with probability p; on a sunny day, component i will function with probability q for $i = 1, \cdots n$. It will rain tomorrow with probability α. Calculate the conditional expected number of components given that it will rain tomorrow.

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Suppose X_1 and X_2 are independent binomial random variables with respective parameters (n_1, p) and (n_2, p), calculate the conditional probability mass function of X_1 given that $X_1 + X_2 = m$.

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$$f(x,y) = \begin{cases} cxy(2-x-y), & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Calculate $E(X|Y=y)$ where $0<y<1$.

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Example

$$f(x,y) = \begin{cases} 4y(x-y)e^{-(x+y)}, & 0 < x < 1,\ 0 \le y \le x \\ 0, & \text{otherwise} \end{cases}$$

Calculate $E(X|Y=y)$.

Computing Expectations by Conditioning

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This can be derived from expectation of function of a random variable. Yet, the proof in the book is quite nice.

Computing Expectations by Conditioning

Example

Suppose that the expected number of accidents per week at an industrial plant is four. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 2. Assume also that the number of workers injured in each accident is independent of the number of accidents that occur. What is the expected number of injuries during a week?

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Additional Homework Question

Suppose that a die is rolled until we observe a 6-4 (in that order). What is the expected number of 4s rolled?

Computing Variance by Conditioning

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$$Var(X) = E[Var[X|Y]] + Var[E[X|Y]]$$

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Probability density functions of X and Y are $f_X(x)$ and $f_Y(y)$ respectively. Calculate $P(X < Y)$.

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X_m is the time (in hours) that Mark goes to restroom. $X_m \sim \text{UNIFORM}(0, 1)$
X_d is the time (in hours) that Dean goes to restroom. $X_d \sim \text{UNIFORM}(0.5, 2.5)$
They both stayed for a duration of 15 minutes. What is the probability that they see each other?

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Example

$X \sim \text{UNIFORM}(0, 1)$ and $Y \sim \text{UNIFORM}(0, 1)$. What is the distribution of $X + Y$?

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Read: Polya's Urn Model and Uniform Priors.

Computing Probabilities by Conditioning

Example Suppose that the number of people who visit a yoga studio each day is a Poisson random variable with mean λ. Suppose further that each person who visits is, independently, female with probability p or male with probability $1 - p$. Find the joint probability that exactly n women and m men visit the academy today.

Homework Assignment

1. The joint density of X and Y is given by $f(x, y) = (e^{-x/y} e^{-y})/y, 0 < x < \infty, 0 < y < \infty$. Find $E[X|Y = y]$.
2. A coin that comes up heads with probability p is continually flipped until the pattern T, T, H appears. (That is, you stop flipping when the most recent flip lands heads, and the two immediately preceding it lands tails.) Let X denote the number of flips made, and find $E[X]$. Let Y denote the number of heads observed, and find $E[Y]$.
3. Two players take turns shooting at a target, with each shot by player i hitting the target with probability $p_i, i = 1, 2$. Shooting ends when two consecutive shots hit the target. Let μ_i denote the mean number of shots taken when player i shoots first, $i = 1, 2$. Find μ_1 and μ_2.
4. $X \sim$ UNIFORM$(0, 1)$ and $Y \sim$ UNIFORM$(0, 2)$. What is the distribution of $X + Y$?

Homework Assignment

5. A basketball player has a rating of 1 to k, where the probability of i rating is p_i with $\sum_{i=1}^{k} p_i = 1$. The number of points scored by an i rating player is Poisson distributed with rate $\lambda_i, i = 1, \ldots, k$. Given that the player scored n points in her first game, what is the expected points she will score in her second game?
6. Independent trials, each resulting in success with probability p, are performed. Find the expected number of trials needed for there to have been both at least n successes and at least m failures. (Hint: Condition on the result of the first $n + m$ trials.)

Stochastic Processes

Two important types of *stochastic processes* we will see are

- Markov chains
- Queueing models

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When T is a countable set, the stochastic process is said to be a **discrete-time process**
When T is an interval of the real line, the stochastic process is said to be a **continuous-time process**

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We will use $X(t)$ and X_t interchangeably.

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  Number of people that can be packed in a car.
- Infinite state space: If X_t may take an infinite number of different values, then the stochastic process is said to have an *infinite state space*.
  Fuel level of a car.

Examples

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- Suppose weather changes from day to day. If the weather is dry today, then it will be dry tomorrow with probability 0.8. If it is raining today, then it will be rainy tomorrow with probability 0.4. Starting on some initial day (labeled as zero), the weather is observed on each day $t = 0, 1, \ldots$.

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- At $t = 0$, we have \$2. At times $t = 1, 2, \ldots$ we play a game in which we bet \$1. With probability p we win the game (our fortune increases by \$1) and with probability $1 - p$ we lose the game (our fortune decreases by \$1).

Examples

Indicate if the following are discrete or continuous time and if state space is finite or infinite.

- Suppose weather changes from day to day. If the weather is dry today, then it will be dry tomorrow with probability 0.8. If it is raining today, then it will be rainy tomorrow with probability 0.4. Starting on some initial day (labeled as zero), the weather is observed on each day $t = 0, 1, \ldots$.
- At $t = 0$, we have \$2. At times $t = 1, 2, \ldots$ we play a game in which we bet \$1. With probability p we win the game (our fortune increases by \$1) and with probability $1 - p$ we lose the game (our fortune decreases by \$1).
- The height of a person in centimeters.

Examples

- A camera store stocks a particular model to be ordered weekly. Let D_t be the demand in week t. Demand is observed during weekdays only. Note that D_t includes lost sales if there is no camera in stock. Inventory management policy is the following: Each Saturday the owner places an order of 3 units if no camera is left in stock. Otherwise, no order is placed. Orders arrive Monday morning before the store is opened. We are interested in modeling the stock level of the store over time.

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  - What if we are interested in the number of lost customers (e.g., to incur a penalty per lost customer) as well?

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In other words, the **next state depends only on the current state**, and is *independent from all past states*. This holds for all points in time.

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Example: Suppose weather changes from day to day. If the weather is dry today, then it will be dry tomorrow with probability 0.8. If it is raining today, then it will be rainy tomorrow with probability 0.4. Starting on some initial day (labeled as zero), the weather is observed on each day $t = 0, 1, \ldots$.

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What is the probability that it will rain on 11th, given that it rained on 10th, dry on 9th, dry on 8th?

Markov Chains

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The conditional probabilities $P(X_{t+1} = j | X_t = i)$ are called the *one-step* transition probabilities.

Stationarity

If, in addition, for each i and j,

$$P(X_{t+1} = j | X_t = i) = P(X_1 = j | X_0 = i)$$

for all $t = 0, 1, 2, \ldots$, then we say that the (one-step) transition probabilities are stationary.

Discrete Time Markov Chains

Example: GAMBLER'S RUIN PROBLEM

- At $t = 0$ we have \$2
- At $t = 1, 2, \ldots$ we play a game in which we bet \$1
- with probability of p we win the game (our fortune increases by \$1)
- with probability of $1 - p$ we lose the game (our fortune decreases by \$1)
- we quit the game if our capital is increased to \$4 or drops to \$0

Show the one-step transition probability matrix. Draw the states and show the transitions.

Discrete Time Markov Chains

n-step transition probability $P_{ij}^{(n)}$: conditional probability that system will be in state j after exactly n steps (time steps), given that it starts in state i at any time t

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Question: Prove that n-step transition probabilities are stationary if one-step transition probabilities are stationarity.

Discrete Time Markov Chains

n-step transition probabilities

$$P^{(n)} = \left\| \begin{matrix} P_{00}^{(n)} & P_{01}^{(n)} & \dots & P_{0M}^{(n)} \\ \vdots & & & \vdots \\ P_{M0}^{(n)} & \dots & \dots & P_{MM}^{(n)} \end{matrix} \right\| \rightarrow \text{n-step transition probability matrix}$$

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$$P_{ij}^{(n)} = \sum_{k=0}^{M} P_{ik}^{(m)} P_{kj}^{(n-m)}, \quad \forall i, j, \ \forall m, n$$

Classification of States

- two states i and j

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$$\lim_{n \to \infty} \sum_{l=1}^{n} P_{ij}^{(l)} = 1$$

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  - if state i communicates with state j, then state j communicates with state i

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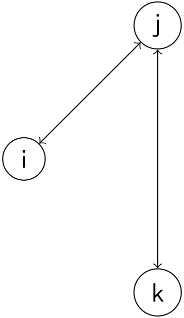
If state i communicates with state j, and state j communicates with state k, then state i communicates with state k.



Prove this statement.

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E.g., are the weather and gambler's ruin MCs irreducible?

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E.g., which states (classes) are recurrent/transient in the weather and gambler's ruin MCs?

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Prove the following:

- State i transient iff $\exists$ state j accessible from i, where i is not accessible from j.
- In a finite-state Markov Chain, not all states can be transient.

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E.g., what is the period for each state (class) in the weather and gambler's ruin MCs?

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- In a finite-state MC, recurrent states that are aperiodic are called **ergodic**
- A MC is ergodic if all states are ergodic
- This is a key property for the existence of *steady-state probabilities*

Long-run Properties

Limiting Probabilities

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What would you observe if you find $P^{(n)}$ where n is large for the weather example?

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How many equations/variables are there in this system?

Limiting Probabilities

Example:

An individual possesses an umbrella that he employs in going from his home to office, and vice versa. If he is at home (the office) at the beginning (end) of a day and it is raining, then he will take his umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (end) of a day with probability p.

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What if he has 3 umbrellas?

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2. Stationary probabilities

If $P(X_0 = j) = \pi_j \quad \forall j$, then the probability of finding the process in state j after $n = 1, 2, \ldots$ transition is also π_j

Interpretation of Limiting Probabilities

3. Equilibrium probabilities

$$\begin{tikzcd} i \arrow[r] & j \arrow[r] & k \end{tikzcd}$$

probability of entering state j = probability of leaving state j

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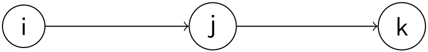

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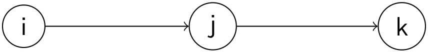

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Example: $$\left\|\begin{array}{cc|ccc} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.8 & 0.2 \end{array}\right\|$$

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- What if we don't have an irreducible MC but have more than one recurrent class?

Example: $$\left\|\begin{array}{cc|ccc} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.8 & 0.2 \end{array}\right\|$$

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i.e., $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} = \pi_j$ always exists

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Example: Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Specifically, suppose that if it has rained for the past two days, then it will rain tomorrow with probability $3/4$; if it rained today but not yesterday, then it will rain tomorrow with probability $1/2$; if it rained yesterday but not today, then it will rain tomorrow with probability $1/3$; if it has not rained in the past two days, then it will rain tomorrow with probability $1/4$. Each morning it rains, you spend \$4 for your Caffè Latte. If it doesn't rain, you spend \$3 for your orange juice instead. What is your average monthly expense on your morning beverage?

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Example:

Given the one step transition matrix for states 0 to 3 for the inventory example as $P = \begin{Vmatrix} 1/10 & 2/10 & 4/10 & 3/10 \\ 2/3 & 1/3 & 0 & 0 \\ 1/4 & 3/8 & 3/8 & 0 \\ 1/10 & 2/10 & 4/10 & 3/10 \end{Vmatrix}$, find the expected time until we place an order given that we start with 3 units.

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Example: In the inventory example, given that an order is placed this week, how many weeks does it take for us on average to place the next order?

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  Note that this figure will always be greater than or equal to 1 as the process already enters the state, thus will definitely spend at least 1 time unit.

Expected Time Spent in Set of States

Example:

Sales can be in one of the 4 states, 0 being the best, 3 the worst.

$$P = \left\| \begin{array}{cccc} 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \end{array} \right\|$$

Assume sales are considered "FAST" when states 0 and 1 are visited, and "SLOW" when in 2 and 3.

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<u>Determine</u>

1. the rate at which sales fo from FAST to SLOW (rate of economic breakdowns)

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Determine

1. the rate at which sales fo from FAST to SLOW (rate of economic breakdowns)
2. the average length of time sales remain SLOW when they go SLOW

Expected Time Spent in Set of States

Example:
Sales can be in one of the 4 states, 0 being the best, 3 the worst.

$$P = \left\| \begin{array}{cccc} 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \end{array} \right\|$$

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If k is an absorbing state and the process starts in state i, the probability of ever going into state k is called *probability of absorption* f_{ik} into state k given that the system starts in state i.

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i.e., Suppose X_t denotes the money on hand for player A. What is f_{02}?

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Example: 2 players A and B have \$2 each. They bet \$1 at a time and they keep playing until one of them goes broke. A wins each bet with probability $1/3$. What is the probability that player A loses the game? What is the expected number of rounds B bets his/her last dollar?

A Random Walk Model

A MC model whose state space is given by integers $i = 0, \pm 1, \pm 2, \ldots$ is said to be a *random walk* if for some number $0 < p < 1$

$$P_{i,i+1} = p, \quad i = 0, \pm 1, \pm 2, \ldots$$

$$P_{i,i-1} = 1 - p, \quad i = 0, \pm 1, \pm 2, \ldots$$

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Example: Consider a random walk with states $0, 1, \ldots, M$ and transition probabilities $P_{i,i+1} = \alpha_i = 1 - P_{i,i-1}$, $i = 1, \ldots, M-1$, $P_{0,1} = \alpha_0 = 1 - P_{0,0}$, $P_{M,M} = \alpha_M = 1 - P_{M,M-1}$. Is this chain time reversible?

Homework Assignment

1. A transition probability matrix P is said to be *doubly stochastic* if the sum over each column equals one; that is, $\sum_i P_{ij} = 1, \forall j$. If such a chain is irreducible and aperiodic and consists of $M+1$ states $0, 1, \ldots, M$, show that the limiting probabilities are given by $\pi_j = 1/(M+1)$, $j = 0, 1, \ldots, M$.
2. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of 4 pairs of running shoes, what proportion of the time does he run barefooted?
3. A flea moves around the vertices of a square in the following manner: Whenever it is at vertex i it moves to its clockwise neighbor vertex with probability p_i and to the counterclockwise neighbor with probability $q_i = 1 - p_i, i = 1, 2, 3, 4$. Find the proportion of time that the flea is at each of the vertices. How often does the flea make a counterclockwise move which is then followed by five consecutive clockwise moves?

Homework Assignment

4. Three out of every four trucks on the road are followed by a car and one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

5. A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. Show that this Markov chain is time reversible.

6. A taxi driver provides service in two zones of a city. Fares picked up in zone A will have destinations in zone A with probability 0.6 or in zone B with probability 0.4. Fares picked up in zone B will have destinations in zone A with probability 0.3 or in zone B with probability 0.7. The driver's expected profit for a trip entirely in zone A is 6; for a trip entirely in zone B is 8; and for a trip that involves both zones is 12. Find the taxi driver's average profit per trip.

Homework Assignment

7. A group of n processors is arranged in an ordered list. When a job arrives, the first processor in line attempts it; if it is unsuccessful, then the next in line tries it; if it too is unsuccessful, then the next in line tries it, and so on. When the job is successfully processed or after all processors have been unsuccessful, the job leaves the system. At this point we are allowed to reorder the processors, and a new job appears. Suppose that we use the one-closer reordering rule, which moves the processor that was successful one closer to the front of the line by interchanging its position with the one in front of it. If all processors were unsuccessful (or if the processor in the first position was successful), then the ordering remains the same. Suppose that each time processor i attempts a job then, independently of anything else, it is successful with probability p_i. Define an appropriate Markov chain to analyze this model. Show that this Markov chain is time reversible. Find the long-run probabilities.
8. On a chessboard compute the expected number of plays it takes a knight, starting in one of the four corners of the chessboard, to return to its initial position if we assume that at each play it is equally likely to choose any of its legal moves. (No other pieces are on the board.) Hint: Make use of Ex. 4.36.

MDP Definition

- Infinite horizon probabilistic dynamic programming problems are called Markov decision processes (or MDPs).

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  Markovian Property
- During a period in which the state is i and an action a is chosen, an expected re- ward of r_{ia} is received.

Example

At the beginning of each week, a machine is in one of four conditions (states): excellent (E), good (G), average (A), or bad (B). The weekly revenue earned by a machine in each type of condition is as follows: excellent, \$100; good, \$80; average, \$50; bad, \$10. After observing the condition of a machine at the beginning of the week, we have the option of instantaneously replacing it with an excellent machine, which costs \$200. The quality of a machine deteriorates over time, as shown below. For this situation, determine the state space, decision (action) sets, transition probabilities, and expected rewards.

	Next week			
Present	E	G	A	B
E	.7	.3	0	0
G	0	.7	.3	0
A	0	0	.6	.4
B	0	0	0	1

Solution

State Space

$\{E, G, A, B\}$

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State Space
{E, G, A, B}

Set of Actions
R: replace at beginning of current period
NR: do not replace during current period

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State Space
{E, G, A, B}

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Transition Probabilities
$P(E|NR, E) = .7$
$P(A|NR, G) = .3$
$P(B|NR, B) = 1$
$P(E|G, R) = P(E|A, R) = P(E|B, R) = .7$
$P(G|G, R) = P(G|A, R) = P(G|B, R) = .3$
$P(A|G, R) = P(A|A, R) = P(A|B, R) = 0$
etc.

Solution

State Space
{E, G, A, B}

Set of Actions
R: replace at beginning of current period
NR: do not replace during current period

Transition Probabilities
$P(E|NR, E) = .7$
$P(A|NR, G) = .3$
$P(B|NR, B) = 1$
$P(E|G, R) = P(E|A, R) = P(E|B, R) = .7$
$P(G|G, R) = P(G|A, R) = P(G|B, R) = .3$
$P(A|G, R) = P(A|A, R) = P(A|B, R) = 0$
etc.

Rewards
$r_{E,NR} = \$100, r_{G,NR} = \$80, r_{A,NR} = \$50, r_{B,NR} = \$10,$
$r_{E,R} = r_{G,R} = r_{A,R} = r_{B,R} = -\$100.$

Solution

$$\max z = 100\pi_{ENR} + 80\pi_{GNR} + 50\pi_{ANR} + 10\pi_{BNR} - 100(\pi_{GR} + \pi_{AR} + \pi_{BR})$$

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$$\pi_{GNR} + \pi_{GR} =$$

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$$\pi_{GNR} + \pi_{GR} = .3(\pi_{GR} + \pi_{AR} + \pi_{BR} + \pi_{ENR})$$

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& \pi_{ENR}, \pi_{GNR}, \pi_{ANR}, \pi_{BNR}, \pi_{GR}, \pi_{AR}, \pi_{BR} \geq 0
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This LP has an optimal solution in which for each state, at most one action has a π value greater than zero. Thus, the optimal solution to this LP will occur for a stationary policy.

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The optimal objective function value for this LP is $z = 60$. The only nonzero decision variables are $p_{ENR} = .35, p_{GNR} = .50, p_{AR} = .15$. Thus, an average of \$60 profit per period can be earned by not replacing an excellent or good machine but replacing an average machine.

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- However, in many stochastic processes the system is observed *continuously* over time, i.e., $t \geq 0$.
- Now, we will discuss properties of continuous stochastic processes with the Markovian property.
- Continuous time Markov chains also provide a link to the **queueing theory**.

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Note: In this course we will be dealing with continuous time Markov chains where

- the state space is finite
- the transition probabilities are stationary

but keep in mind that there is a lot beyond these restrictive assumptions.

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- The next state visited is independent from T_i.
- In other words, a continuous time Markov chain is a continuous process that moves from state to state according to a discrete time Markov chain, but is such that the amount of time spent in each state, before proceeding to the next state, is **exponentially** distributed.

Why Exponential?

Markovian Property

A stochastic process has the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present states) depends only upon the present state, not on the sequence of events that preceded it.

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Memoryless Property of Exponential
Do you remember the proof?

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MEMORYLESS PROPERTY OF EXPONENTIAL
Do you remember the proof?

$$P\{X > s + t | X > t\} = P\{X > s\} \; \forall s, t \geq 0$$

Examples

- Amount of time one spent in a bank is exponentially distributed with mean 10 min. What is the probability that a customer will spend more than 15 min. in the bank? What is the probability that a customer will spend more than 15 min. in the bank given that she is still in the bank after 10 min.?

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- A post office is run by two clerks. Suppose that when Mr. Smith enters the system, he discovers that Mr. Jones is being served by one of the clerks and Mr. Brown by the other. Suppose also that Mr. Smith is told that his service will begin as soon as either Mr. Jones or Mr. Brown leaves. If the amount of time that a clerk spends with a customer is exponentially distributed with mean $1/\lambda$, what is the probability that, of the three customers, Mr. Smith is the last to leave the post office?

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Important Note: Always pay attention to the units of exponential/poisson parameters! Parameters will not always be given explicitly.

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- Note that Poisson process has *stationary increments* and $E[N(t)] = \lambda t$

Key Properties

Next we present 4 key properties related to Exponential distribution / Poisson processes.

Make sure you know how to prove each of those statements

Key 1: Interarrival times of Poisson processes are Exponential

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Key 2: Fractions of Poisson process are Poisson

Consider a Poisson process with rate λ. Suppose each event is of type i independently with probability p_i. Then type i events are **Poisson distributed** with rate λp_i.

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Remember the example: Suppose that the number of people who visit a yoga studio each day is a Poisson random variable with mean λ. Suppose further that each person who visits is, independently, female with probability p or male with probability $1 - p$. Find the joint probability that exactly n women and m men visit the academy today.

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PROBABILITY THAT ONE VARIABLE IS THE MINIMUM OF A SET IS EQUAL TO ITS RATE DIVIDED BY SUM OF ALL RATES.

Example

Suppose you are one of the two clerks in Şok Market. If nobody is in the system, the next customer chooses your register with probability .75. Assume 18 customers arrive the register area in an hour on average with a Poisson distribution. Time it takes you (other clerk) to serve a customer is 10 (5) minutes on average and both service times are exponentially distributed. Finally, suppose both clerks share the same queue (line).

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- When both clerks are busy and there is someone in the queue, what is the expected time that the person in line waits?

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- When both clerks are busy and there is someone in the queue, what is the probability that you are done with your current service, before a female joins the queue? (Suppose each arrival is female with probability 2/3)

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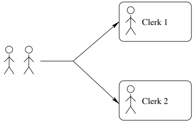
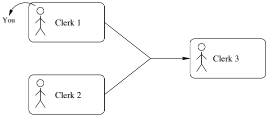
- When both clerks are busy and there is someone in the queue, what is the probability that you are done with your current service, before a female joins the queue? (Suppose each arrival is female with probability 2/3)
- In the long run, what is the fraction of time you are working? (up next)

Homework Assignment

1. In a system, an incoming customer is served by either clerk 1 or 2, where service time for clerk i is exponentially distributed with mean $(i+1)$ hours. Suppose that whichever clerk finishes service first, starts serving a customer from the queue immediately. If both servers are busy and there are two people waiting in the queue, what is the expected time spent to serve these four customers? Suppose that nobody else can enter the system.



Scenario in question 1



Scenario in question 2

2. In a bank, an incoming customer is first served by either clerk 1 or 2, and next by clerk 3. In front of clerk 3, there is a FIFO queue: that is clerk 3 serves the customers in the order they are done with either clerk 1 or 2. Assume service time for clerk i is exponentially distributed with mean $15 \times i$ minutes. If there are a total of 3 customers to be served and you are currently being served by clerk 1, find your expected time spent in this system.

Homework Assignment

3. There is a dog and a cat whose lifetimes are independent exponential random variables with respective rates λ_d and λ_c . One of them has just died. Find the expected additional lifetime of the other pet.
4. A doctor has scheduled two appointments, one at 1 P.M. and the other at 1:30 P.M. The amounts of time that appointments last are independent exponential random variables with mean 30 minutes. Assuming that both patients are on time, find the expected amount of time that the 1:30 appointment spends at the doctor's office.
5. Each entering customer must be served first by server 1, then by server 2, and finally by server 3. The amount of time it takes to be served by server i is an exponential random variable with rate $\mu_i, i = 1, 2, 3$. Suppose you enter the system when it contains a single customer who is being served by server 3. Find the probability that server 3 will still be busy when you move over to server 2. Find the probability that server 3 will still be busy when you move over to server 3. Find the expected amount of time that you spend in the system. (Whenever you encounter a busy server, you must wait for the service in progress to end before you can enter service.)

Homework Assignment

6. Suppose X_1 and X_2 are two exponentially distributed random variables. Can you find $\max(X_1, X_2)$?
7. In a sequential setup, where you are behind someone, would you rather have the faster or slower server first?

Kendall's Notation

- Example: $M/M/1/K/N/\text{FIFO}$
  - M denotes arrivals (M: Markovian (Poisson), G: General Distribution, E, etc.)
  - M denotes service (M: Markovian (Exponential service), D: Deterministic, G, E, etc.)
  - 1 denotes # of servers (1,2,3..)
  - K denotes system capacity (default ∞)
  - N denotes calling population (default ∞)
  - FIFO denotes queue discipline (SIRO, LIFO, etc.)

Limiting Probabilities for Continuous Time Markov Chains

$$\text{RATE IN} = \text{RATE OUT}$$

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Consider an $M/M/1$ system. Assume λ (μ) is the arrival (service) rate.

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It is intuitive that λ must be less than μ for limiting probabilities to exist. If $\lambda > \mu$, the queue size will go to infinity. Don't worry about the exceptional case of $\lambda = \mu$ (i.e., symmetric random walk), which is *null recurrent* and thus has no limiting probabilities.

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Assuming $\lambda < \mu$, what is the utilization of the server in the long run?
What is the expected number of people in the system?

Example

Suppose you are one of the two clerks in Şok Market. If nobody is in the system, the next customer chooses your register with probability 2/3. Assume 9 customers arrive the register area in an hour on average with a Poisson distribution. Time it takes you (other clerk) to serve a customer is 10 (5) minutes on average and both service times are exponentially distributed. Finally, suppose both clerks share the same queue (line).

- In the long run, what is the fraction of time you are working?

Example

A job shop consists of three machines and two repairmen. The amount of time a machine works before breaking down is exponentially distributed with mean 10. If the amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8, then what is the average number of machines not in use?

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Example

Consider a two-server parallel queuing system where customers arrive according to a Poisson process with rate 4 per minute and where the service times are exponential with mean 30 seconds. Moreover, suppose that arrivals finding both servers busy and one person in the queue immediately depart without receiving any service (such cost is said to be lost). Assume that a server charges \$5/min. What is the rate charged by this system?

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Example

Camera store inventory — a continuous representation

- Suppose $D \sim \text{Poisson}(2/\text{week})$
- Inventory management policy is a continuous review policy
  - The owner places an order of three units if no camera is left in stock
  - That is a (Q, R) policy with $Q = 3$ & $R = 0$.
  - Assume zero lead time
- We are interested in modeling the stock level at the store over time
- Assume that the inventory holding cost is \$2/item/week and there is a fixed ordering cost of \$10/order.
- Find the expected monthly inventory holding cost and ordering cost.

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- Find the expected monthly inventory holding cost and ordering cost.
- What would you change if $R = 1$ and lead time is exponentially distributed with mean one day? Assume you allow backorders and place an order of 3 items whenever your inventory position hits 1.

Homework Assignment

1. What is the expected number of people in the system for an $M/M/3$ system?
2. Consider a general birth and death process with birth rates λ and death rates μ. Starting with a population of size i, what is the expected time to reach a population of size j, $j > i$?
3. Each time a machine is repaired it remains up for an exponentially distributed time with rate λ. Failures can be of two types: repair for a type 1 (2) failure is exponential with rate μ_1 (μ_2). Each failure is, independent of the failure time, a type 1 failure with probability p and a type 2 failure with probability $1 - p$. What proportion of time is the machine down due to a type 1 and 2 failure? What proportion of time is it up?
4. Customers arrive at a two-server station in accordance with a Poisson process having rate λ. Upon arriving, they join a single queue. Whenever a server completes a service, the person first in line enters service. The service times of server i are exponential with rate $\mu_i, i = 1, 2$, where $\mu_1 + \mu_2 > \lambda$. An arrival finding both servers free is equally likely to go to either one. Define an appropriate CTMC for this model, show it is time reversible, and find the limiting probabilities. What should we change regarding an arrival finding both servers free so that this chain is not time reversible?

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- # of servers: 2
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What is the fraction of time server 1 is idle?
What is the expected number of people in the system?
What is the expected time spent in the system for a person?

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Fundamental Metrics

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$$\lambda_a : \textit{effective arrival rate} = \lim_{t\to\infty} \frac{N(t)}{t}$$

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Queueing Theory

Example:

- Consider an $M/M/1$ system
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- $\mu = 3$

Calculate expected # of people in system, expected # of people in queue, expected # of people in service, expected time spent in system, expected time spent in queue, and expected time spent in service.

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What would you change if it was an $M/M/1/K$ system? (aside from limiting probability calculation: sum of finite series)

Steady-State Probabilities

$$\pi_n : \lim_{t \to \infty} P\{X(t) = n\}$$

- a_n: proportion of customers to find n in the system
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$$a_n = \frac{r_{n\to n+1}}{\text{arrival rate}}, \quad d_n = \frac{r_{n+1\to n}}{\text{departure rate}}$$

Steady-State Probabilities

$$\pi_n : \lim_{t\to\infty} P\{X(t) = n\}$$

- a_n: proportion of customers to find n in the system
- d_n: proportion of customers leaving behind n in the system

In any system in which customers arrive and depart one at a time, $a_n = d_n$
Proof: Using time reversibility, rate of $n \to n+1$ = rate of $n+1 \to n$

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PASTA Property

Poisson arrivals see time averages, $\pi_i = a_i = d_i$, $\forall i$

I use π to indicate limiting probabilities but the book uses P for CTMCs.

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- Average length of a busy period: $E[B]$

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Read *Shoe Shine Shop* example in Section 8.3.

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As service rates are supposed to be larger than arrival rates, departure rate from a subsystem is equal to its arrival rate.

Open Network of Queueing System

Example:

Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates of 1 and 2 are respectively 8 and 10. A customer upon completion of service at server 1 is equally likely to go to server 2 or to leave the system whereas a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise. Determine the limiting probabilities, L and W.